Informatique et preuve

Une brève histoire du raisonnement automatisé

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Université catholique de Louvain

Séminaire *fondements et notions fondamentales* – 12 mars 2012

Replacing Scholars by Programs?

From  to 

?

Paul Erdõs

HAL 9000

Computer Proofs?

- Can “creativity” be “automated”?

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- . . . Reasoning **can** be reduced to computation (to some extent)

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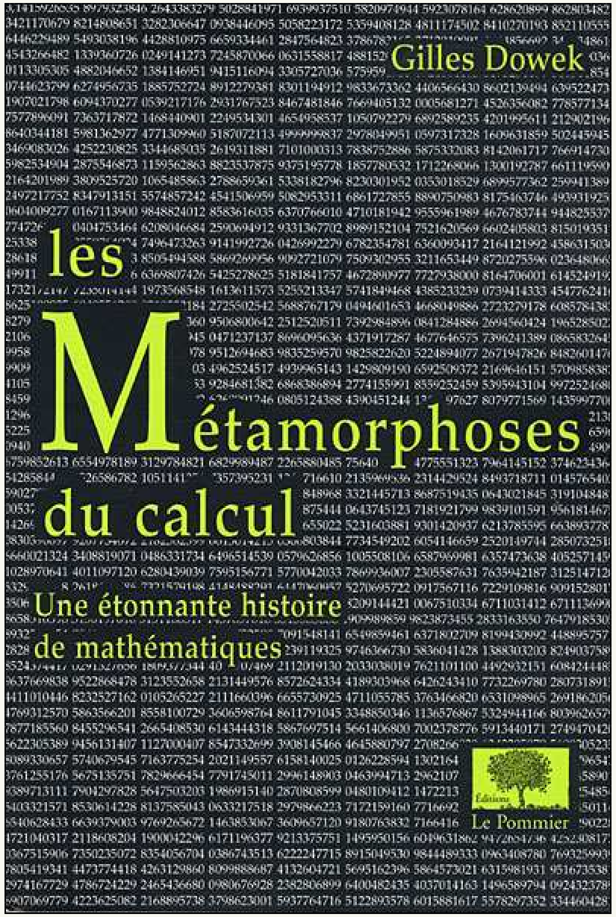
- **What I teach**
  - Beginner programming (Java), system modelling and analysis,
  - **(automated) program proofs**, **automated reasoning**

Inspiring Reading

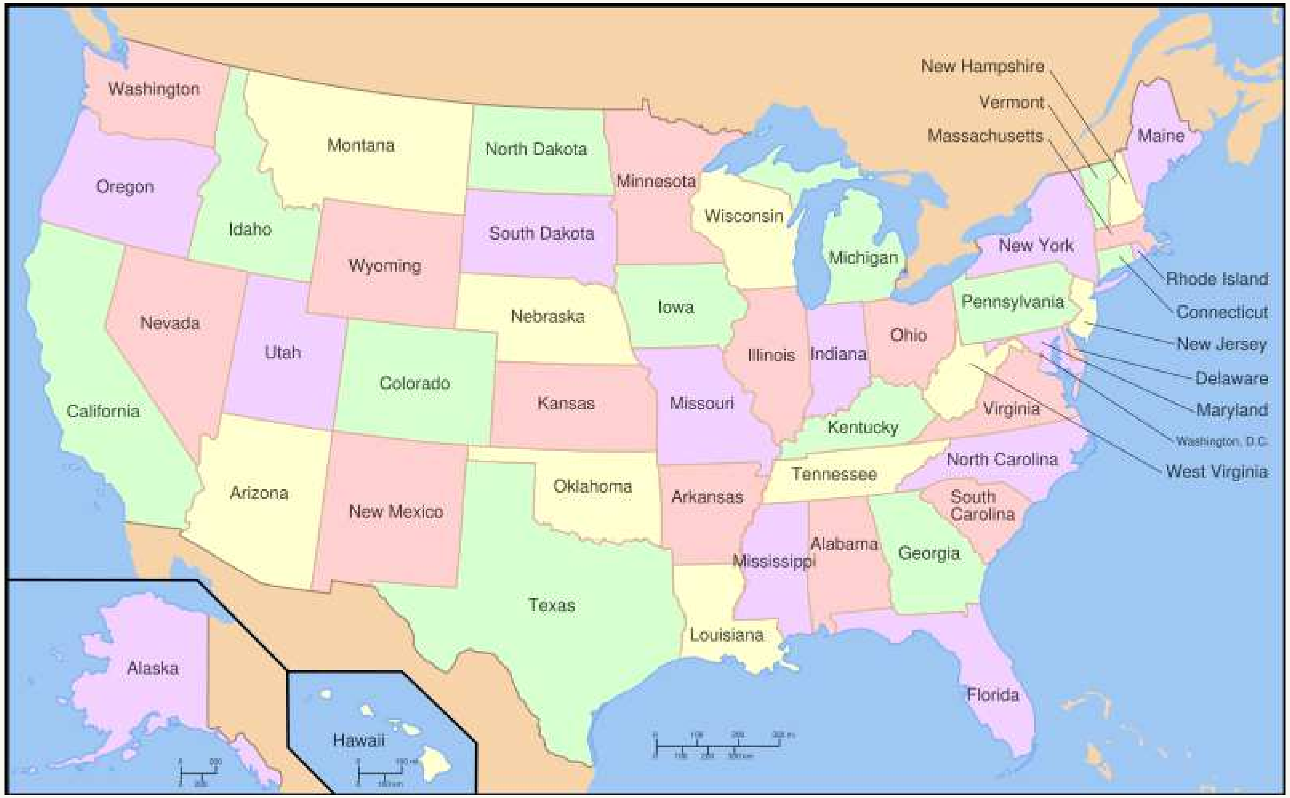


Gilles Dowek
Les métamorphoses du calcul
Une étonnante histoire de mathématiques
Le Pommier, 2007

Contents

AR Examples

The Four Colour Theorem



- *The vertices of every planar graph can be colored with at most four colors so that no two adjacent vertices receive the same color*
- Or equivalently, *any map may be colored using no more than four colors in such a way that no two adjacent regions receive the same color*

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- Proof in **Coq** in 2004 (*Werner, Gonthier*)
  - General-purpose theorem prover
  - Still need to **trust Coq**...

Robbins Algebra are Boolean

- **Robbins algebra**: $(A, \vee, \neg)$ satisfying
  $a \vee (b \vee c) = (a \vee b) \vee c$ (associativity)
  $a \vee b = b \vee a$ (commutativity)
  $\neg(\neg(a \vee b) \vee \neg(a \vee \neg b)) = a$ (Robbins's axiom)

- **Boolean algebra**: $(A, \vee, \wedge, \neg, 0, 1)$ satisfying
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  $a \vee b = b \vee a$ (commutativity)
  $a \vee (a \wedge b) = a$ (absorption)
  $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ (distributivity)
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  …and their duals wrt. $\wedge/\vee$, $0/1$

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- **Conjecture**: all Robbins algebra are Boolean

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  - using EQP = automated prover for equational logic
  - found proof of the missing lemma
  - after 14 attempts totaling five weeks of CPU time

Paris Métro Ligne 14

- Platform screen doors control software
  - Starting/stopping trains, opening/closing train and platform doors
  - Parts on-board, on wayside, at control center

Paris Métro Ligne 14: Proof

T. Lecomte, T. Servat, G. Pouzancre. Formal Methods in Satefy Critical Railway Systems. SBMF 2007.

- Safety-critical code written in B
  - Includes **formal safety properties**
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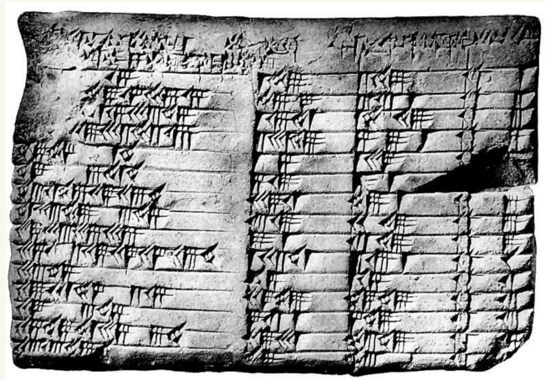
- Seems to work!
  - **No bug found** after 9 years of operation

Before AR

The Early Days



- Mesopotamia, since 2500 BC
  - Add, multiply, divide, area of rectangles, triangles, disks, . . .
  - With **given** numbers: **computing**

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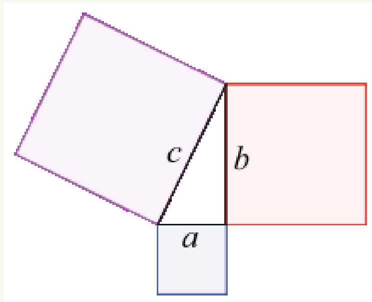

- Pythagoras, 500 BC:
  - **For all** rectangle triangles (a, b, c): $a^2 + b^2 = c^2$
  - **Infinitely many** (a, b, c): **reasoning**

(images from Wikipedia)

And Then Logics

- Aristote, 350 BC:

All men are mortal.
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- Seen as **philosophy**, not mathematics!
  - Euclid's Elements did not (explicitly) use them!
  - Too crude: needs functions, predicates

Reasoning as Computing?

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  - “Reason [. . .] is nothing but reckoning [= calculating]” (T. Hobbes, 1651)

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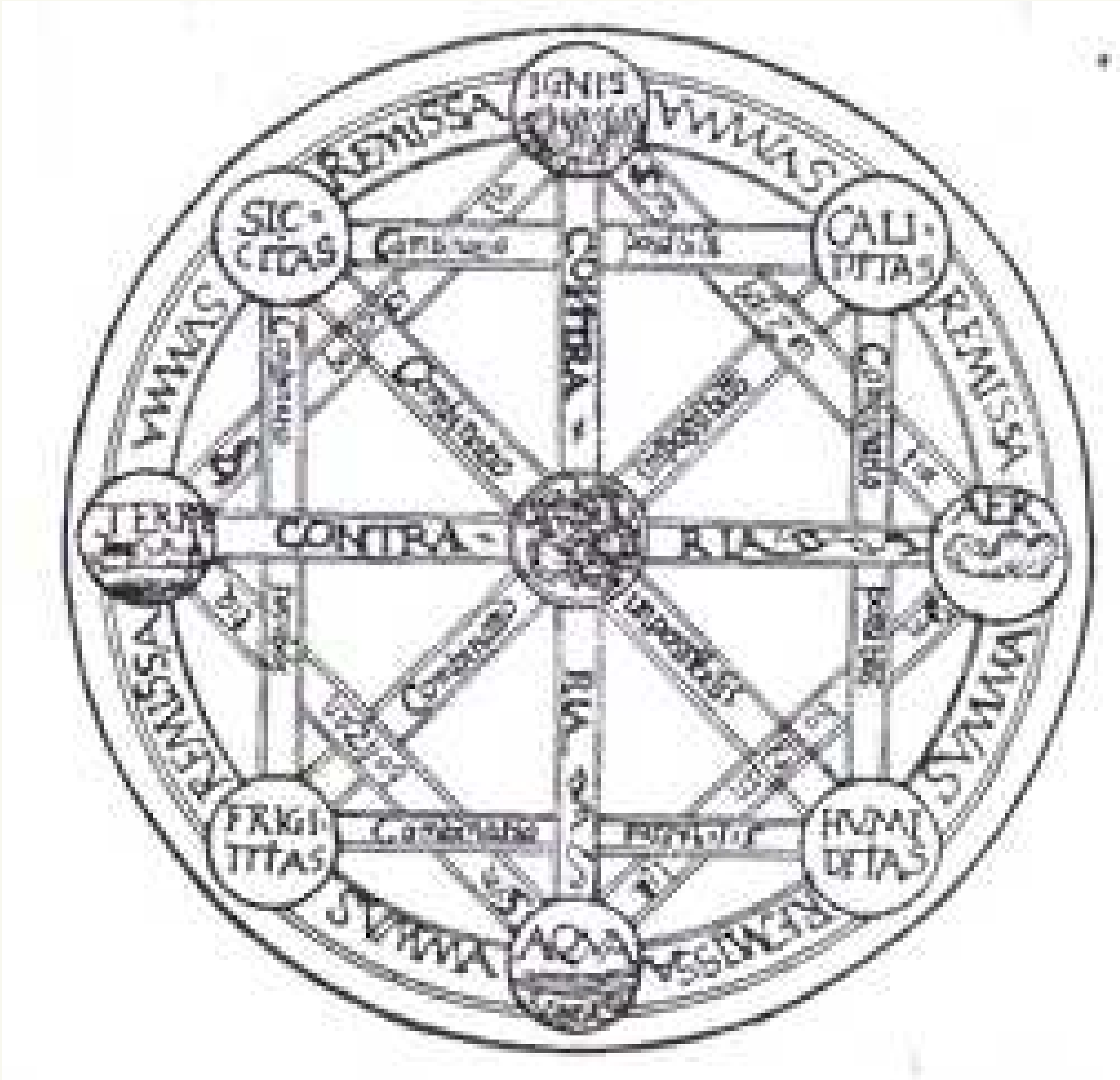
  - "Reason [. . .] is nothing but reckoning [= calculating]" (T. Hobbes, 1651)
- *Characteristica Universalis* (Leibniz, 1646–1716)
  - An (unrealized) universal **language** to express mathematical, scientific, and philosophic concepts
  - *Calculus ratiocinator* (calculus of reasoning): an (unrealized) universal logical **calculation**



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Formalizing Logics

- *Calculus of logic* (Boole, 1815–1864)
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- *Principia Mathematica* (Whitehead and Russell, 1910)
  - **Type** theory
  - Formal foundations of **mathematics**

Frege's Begriffsschrift

Basic concept	Frege's notation	Modern notations
Judging	$\vdash A$, $\Vdash A$	$p(A) = 1$ $p(A) = i$
Negation	A	$\neg A$; $\sim A$
Conditional (implication)	A B	$B \rightarrow A$ $B \supset A$
Universal quantification	$\mathfrak{u}$ $\Phi(\mathfrak{u})$	$\forall y$: $\Phi(y)$
Existential quantification	$\mathfrak{u}$ $\Phi(\mathfrak{u})$	$\exists y$: $\Phi(y)$
Content identity (equal sign)	$A \equiv B$	$A = B$

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Reasoning as Computing. . . or Not?

- Hilbert's program (Hilbert, 1922)
  - (Science program, not computer!)
  - Goal: **formalize** all of mathematics
  - Goal: **prove** completeness, consistency, . . .
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- The incompleteness theorems (Gödel, 1931)
  - Any "rich enough" formal system is **incomplete**
  - i.e. some valid statements cannot be proven
  - Essential **limit** to Hilbert's goal

Deciding is Computing

- Formalization of computation = **decidability**
  - . . . before creation of computers!
  - **Turing machines** (Turing, 1936)
  - λ**-calculus** (Church, 1936)
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- Then came the **computers** (1940's, WWII)
  - ... and the first attempts to **compute proofs**
  - **Artificial intelligence** (McCarthy, 1956)
  - Lisp (1956), Prolog (1972)

The AR Problem

What's **logic**?

- **Facts**: logic **formulae** ϕ (syntax)

$$\forall a, b, c, n \in \mathbb{N} : n \geq 3 \Rightarrow a^n + b^n \neq c^n$$

- **Reasoning**: logic **proofs** $\phi_1, \ldots, \phi_n \vdash \phi$
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- **Proof = Rules + Strategy = Computing + Reasoning**

Models

What's a **useful** logic?

- **Means** something: **interpretations** M (aka models)
  - Propositions, predicates, functions, sets, numbers, programs, ...
  - **Semantics**: $M \models \phi$ if ϕ is true in/about/for M
  - **Consequence**: $\phi_1, \ldots, \phi_n \models \phi$
  - **Validity**: $Ax \models \phi$
  - **Satisfiability**: $Ax \not\models \neg\phi$
- Reasons **properly**
  - **Soundness**: all **proofs** are **valid**

$$Ax \vdash \phi \quad \Rightarrow \quad Ax \models \phi$$

  - **Completeness**: all **valid facts** can be **proven**

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What's **deciding** a problem?

- **Computing** a **yes-or-no** answer to (any instance of) the problem
- Some things are **undecidable**
  - Does a program terminate?
  - Is a (context-free) grammar unambiguous?
  - Does a Diophantine equation have solutions?
  - **Is a logic formula valid**? (Entscheidungsproblem)

Computing Proofs

- Proofs systems can be used to **enumerate** proofs
  - E.g.: all proofs of length 0 (axioms), then length 1, etc.
  - Fair: will find a proof if there is one. . .
  - . . . but will go forever if there isn't
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- We have at least a **semi-decision** procedure
  (for **theorems** at least, for **validity** if **complete**)
- Common approaches
  - Reduce formulae to **normal forms** (easier for computing)
  - Part of the theory "built-in" the method (e.g. equality), the rest provided as ordinary formulae Ax
  - Proof by **refutation**: (un)**satisfiability** of $Ax \wedge \neg\phi$

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  - Arithmetics on **reals** is **decidable**!
- Many **quantifier-free fragments** are **decidable**
  - Enough for many applications

Decidability and Complexity of Some Theories

Theory	full	CQFF
propositional	NP-comp.	$\Theta(n)$
first-order	no	$\Theta(n)$
equality (uninterpreted fct.)	no	$O(n \log n)$
$\mathbb{N}, +, \times$ (Peano)	no	no
$\mathbb{N}, +$ (Pressburger)	$O(2^{2^{2^{kn}}})$	NP-comp.
$\mathbb{R}, +, \times$	$O(2^{2^{kn}})$	$O(2^{2^{kn}})$
$\mathbb{R}, +$ (or $\mathbb{Q}, +$)	$O(2^{2^{kn}})$	PTIME
recursive data structures	no	$O(n \log n)$
acyclic recursive data struct.	not elementary	$\Theta(n)$
arrays	no	NP-comp.

(CQFF = **conjunctive** quantifier-free formulae)

Using Computed Proofs

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- **Synthesizing** artifacts
  - Constructive proof of $\exists x.\phi(x)$

AR Milestones

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  - Decision algorithm for first-order formulae over $(\mathbb{N}, +)$
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  - Very inefficient! ($O(2^{2^{2^{cn}}})$)

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- Along the same lines:
  - Decision algorithm for $(\mathbb{N}, \times)$ (Skolem 1930)
  - Decision algorithm for $(\mathbb{R}, +, \times)$ (Tarski 1931)
  - NB: Euclidean geometry reducible to $(\mathbb{R}, +, \times)$
  - NB: $(\mathbb{N}, +, \times)$ (Peano) is not decidable (Gödel 1931)

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  - (though propositional logic is decidable!)
- **Geometry Machine** (Gelertner 1963)
  - Proofs for elementary geometry
  - Similar approach
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Computer Proofs: First Steps

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- **Human-like** proofs!

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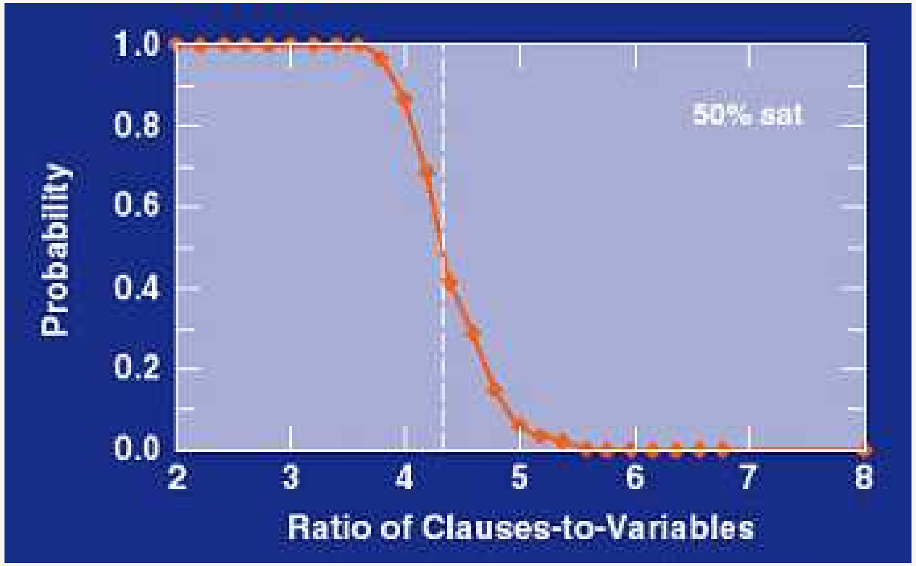
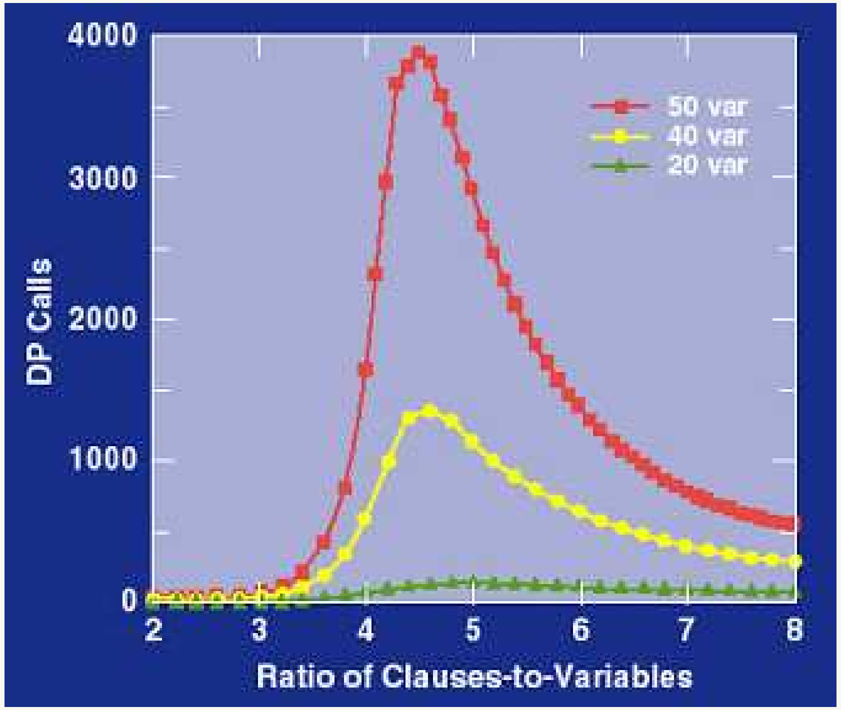
  - Otherwise, choose a literal ℓ and try ℓ then $\neg\ell$ (**case-split**)
- **Computer-like** proofs, not intuitive but efficient!

- DPLL-based SAT solvers **widely used** today
  - Lots of improvements, very efficient implementations
  - Berkmin, Chaff, zChaff, Minisat, . . .
  - Inside many applications
  - Often good performance in practice





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The **Resolution method** (Robinson 1965)

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- This single rule (+ factoring) provides a **complete** proof method for **first-order logic**!

- Limitations of Resolution
  - Clauses, generic rule $\Rightarrow$ inefficient, lacks guidance
  - Need more: equality, numbers, sets, induction, ...

Equational Reasoning

Paramodulation (Robinson, Wos, 1969) another Robinson!

- For proofs with **equational theories**

e.g.
$$0 + x = x$$
$$(x + y) + z = x + (y + z)$$
$$-x + x = 0$$

- Combines resolution and replacing equals by equals

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- **Paramodulation rule**:

$$\frac{\ell_1 \vee \ldots \vee \ell_n \vee \boxed{s = t} \qquad \boxed{\ell'[u]} \vee \ell'_1 \vee \ldots \vee \ell'_m}{\ell_1\sigma \vee \ldots \vee \ell_n\sigma \vee \boxed{\ell'\sigma[t\sigma]} \vee \ell'_1\sigma \vee \ldots \vee \ell'_m\sigma} \quad \boxed{\sigma = mgu(s, u)}$$

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- Used for proof of Robbins conjecture

- **Term Rewriting**
  - Rules $s \to t$ used to reduce (= rewrite) s into t
  - Repeat until irreducible **normal form** $s{\downarrow}$

e.g. $0 + x \to x$
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$\Rightarrow (a + 0) + b$ becomes $a + (0 + b)$ becomes $a + b$

Rewrite Systems

- **Term Rewriting**
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$\Rightarrow (a + 0) + b$ becomes $a + (0 + b)$ becomes $a + b$

- Used for reasoning in **equational theories**
  - Turn equations into rewrite rules
  - If the rules are **convergent**,
    then $s = t$ iff $s{\downarrow}$ and $t{\downarrow}$ are identical
  - **Knuth-Bendix** procedure (1970) for checking convergence
- Also at the core of **functional programming**

Prolog (Colmerauer 1972)

```
ancestor(X,X).
ancestor(X,Z) :- parent(X,Y), ancestor(Y,Z).
parent(albertII,philippe).
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?- ancestor(albertII,X), ancestor(X,elisabeth).
X = albertII
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Logic Programming

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- **Logic clauses** as **program statements**,
  logic reasoning as **program execution**!
- Based on **SLD-resolution** (Kowalski 1973)
  - Resolution specialized on definite clauses
- Prolog adds many programming language features!

Richer Logics

- Higher-Order Logics
  - Functions, sets, relations
- Type systems
  - Numbers, lists, trees, . . .
  - and functions/sets/relations thereof
- Inductive reasoning
- Forces **interactive** approaches = **proof assistants**
  - Most problems are undecidable, huge search spaces
  - Proof tactics and tacticals, proof planning
  - Proof editors and browsers

Some Proof Assistants

- **LCF** (Milner, 1972)
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- **PVS** (Owre, Rushby, Shankar, 1992)
  - Based on sequent calculus

Example: PVS Proof

```
sum_plus :

  |-------
{1}   (FORALL (f: [nat -> nat], g: [nat -> nat], n: nat):
         sum((LAMBDA (n: nat): f(n) + g(n)), n) = sum(f, n) + sum(g, n))

Rule? (skolem!)
Skolemizing,
this simplifies to:
sum_plus :

  |-------
{1}   sum((LAMBDA (n: nat): f!1(n) + g!1(n)), n!1)
          = sum(f!1, n!1) + sum(g!1, n!1)

Rule? (lemma "nat_induction")
Applying nat_induction where
this simplifies to:
sum_plus :

{-1}   (FORALL (p: pred[nat]):
         (p(0) AND (FORALL (j: nat): p(j) IMPLIES p(j + 1)))
             IMPLIES (FORALL (i: nat): p(i)))
  |-------
[1]   sum((LAMBDA (n: nat): f!1(n) + g!1(n)), n!1)
          = sum(f!1, n!1) + sum(g!1, n!1)
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Decision Procedures

- **Automated** decision procedures (DPs) for specific theories
  - **Quantifier-free** fragments
  - (QF) Linear integers/reals $\Rightarrow$ **simplex algorithm**
  - (QF) Polynomials $\Rightarrow$ **Gröbner bases**
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  - Intuition: **proof** = **logic** (SAT) + **theories** (DP)
- Inside many tools: **embedded** automated reasoning

Proving Programs

- Principle: reduce **programs** to **logic**
  - Base case: $\{x \times x > 0\}\; y := x \times x \;\{y > 0\}$
  - Program properties reduce to (first-order) **verification conditions**
  - Prove with standard proof tools (solvers)
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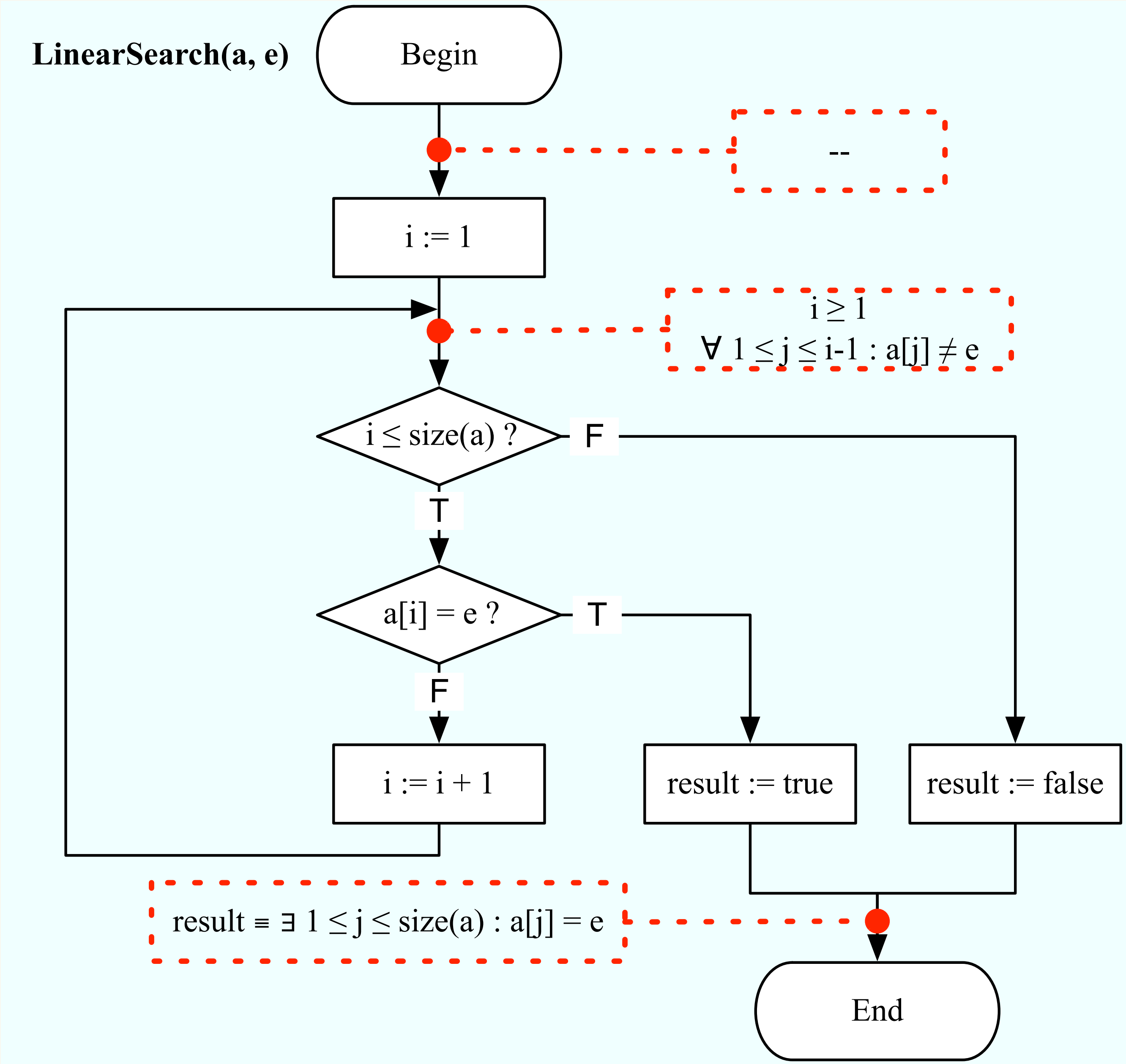
  - Specify **assertions** at connection points
  - Prove that each path preserves the assertions
- Hard problem: loops, recursion, pointers, objects, concurrency, ...
- Lots of conditions to check (thousands) but "easy" proofs
- Example: B method applied to Paris metro line

Example: Inductive Assertions



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  - By exhaustive exploration of M: **semantic** approach
  - Fully automatic! (though computation-intensive)
- Concretely, M = (the state space of) a computer program/system
  - Very large (millions of states), state space explosion
  - Even infinite, with symbolic approaches ($\Rightarrow$ solvers!)
  - Explore all possible **executions**
  - For all parameters, inputs, scheduling, timing
- ϕ = temporal logic

e.g. $\Box\neg(busy_a \wedge busy_b)$
$\Box(send \Rightarrow \Diamond receive)$

AR Perspectives

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- Algorithmic improvements
  - CASC competition (8 divisions, 20+ categories in 2012)

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- Will computer provers someday equal, then surpass humans?
  That is the (weak) AI question!

Bibliography

Bibliography

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